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Modification of the geometry curriculum in relation to the curriculum reform in the light of the Van Hiele levels

Abstract: In this paper the level of geometry education in mathematics education in Hungary is investigated. The relationship between the National Core Curriculum, the Framework Curriculum and the final exam is analyzed from the geometry point of view via the Van Hiele levels as a tool for comparison. It is observed that the geometry problems on the final exams do not follow the level prescribed by the National Core Curriculum. We compare these observations with the results of the Usiskin-test of first year preservice math teacher students.

Titel: Änderung des Geometrie-Lehrplans in Bezug auf die Curriculum-reform im Lichte der Van-Hiele-Niveaustufen

Kurzfassung: In diesem Beitrag wird das Niveau des Lernens von Geometrie im ungarischen Mathematikunterricht untersucht. Die Beziehung zwischen dem Nationalen Kerncurriculum, dem Rahmencurriculum und der Abschlussprüfung wird hinsichtlich des geometrischen Denkens mittels der Van-Hiele-Ebenen als Vergleichsinstrument analysiert. Es zeigt sich, dass die Geometrieaufgaben in den Abschlussprüfungen nicht dem vom Nationalen Kerncurriculum vorgeschriebenen Niveau entsprechen. Wir vergleichen diese Beobachtungen mit den Ergebnissen des Usiskin-Tests von Mathematiklehramtsstudierenden im ersten Studienjahr.

Classification: B30, D70, G10

Keywords: Van Hiele levels, high shool, understanding geometry, development, final exam

Introduction

Years 2014/15 and 2020/21 are milestones in Hungarian mathematical education. In 2014/15 the structure of the final exam was changed, in 2020/21 a new National Core Curriculum (NCC, 2020) was introduced. Geometry is a substantial part of secondary mathematics education as it occupies approximately thirty-five percent of the high school mathematics material, similarly to its proportion in the final exam. Hence it is worth to investigate the relationship between the Van Hiele levels, the new NCC and the new final exam. A similar analysis was done in 2015. In Muzsnay at al. (Muzsnay & al., 2020) the geometrical understanding of Hungarian high-school students is investigated. The aim of their research was to compare high-

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school students' development in geometry, especially to see whether or not students' development in high school geometry is parallel to the requirements of the 2012 National Core Curriculum (NCC, 2012). The coincidence that their survey happened just when the structure of the final exam was changed gives us a chance to make a proper comparison of the geometry curriculum and geometry knowledge of the two curricula. They used the Van Hiele levels to follow the development of geometry high-school knowledge. In particular, they asked if students of grade 12 have achieved level 4, the level of proofs. They measured the Van Hiele level of 342 students from five different high-schools using Usiskin's test (Usiskin, 1982). The survey was held during the 2015/2016 academic year. It was found that the average geometrical understanding of highschool students is below the requirements of the NCC, and this avarage is not changing from grade 9 to grade 12. According to them a possible reason of this phenomenom is that during their high-school years students are prepared to the final exam at the end of grade 12. Although the 30 percent of problems on this test involves geometry, the solution of the problems does not require higher geometrical understanding. They also measured the Van Hiele level of the first year preservice math teacher students.

Since the NCC is changing right now, it would not pay to make a new survey of high school students' geometry knowledge, the coclusions are worthless. Still, we are interested if some change can be observed.

For a reasonable comparison we measured the Van Hiele level of 2nd semester preservice math teachers at Eötvös Loránd University in 2021 and compared it to the same results of 2015 to see if the change of final exam brought a change to the knowledge of geometry. This survey is interesting in itself.

In this paper we go through step by step the required hypothetical geometry knowledge in the Hungarian education system. We introduce the geometry part of the new NCC of 2020 and the corresponding Framework Curricula (FC, 2020). Then we compare the competencies and hypothetical knowledge of students to the Van Hiele levels. By the idea of Muzsnay at al. (Muzsnay & al., 2020) the final exams' geometry problems will be categorized by the Van Hilele levels needed to solve them, and at the end we measure the Van Hiele levels of Hungarian preservice math teachers via the Usiskin test.

The Van Hiele levels and the framework

The NCC contains the mathematics curriculum that should be covered in grades 1-4, 5-8 and 9-12, as well as the competencies to develope. The FC breaks up the NCC and gives a detailed recommendation on the high school material for every two years. The framework suggests even the number of lessons to be spent on a particular topic. Its recommendation allows a bit of movement, flexibility. Then, the schools locally plan the syllabus and the teachers themselves are allowed to even more flexibly adjust this plan to their actual class. After grade 12 comes the final exam that is the entrance exam to the university, as well.

The Van Hiele levels are widely known and understood. In order to make a more condise comparison we give two descriptions of the levels. The first one is the standard and the second one is an interpretation from (Burger, Shaughnessy, 1986) major difference between the two descriptions is that the latter one is in items, has more details and puts some emphasize on what a student cannot know on a particular level. We put the itemized description in Table 1. The general description is the following.

Level 1: Visualization

At this initial stage, students recognize figures only by appearance and they usually think about space only as something that exists around them. Geometric concepts are viewed as undivided, whole entities rather than as having components or attributes. For example, geometric figures are recognized by their whole physical appearance, not by their parts or properties, so the properties of a figure are not detected. A person functioning at this level makes decisions based on perception, not reasoning. On the other hand, they can learn geometric vocabulary, identify specified shapes, reproduce a given figure. However, a person at this stage would not recognize the part of the figures, thus, they cannot identify the properties of these parts.

Level 2: Analyzation

At this level an analysis of geometric concepts begins. For example, students can connect a collection of properties to figures, but at this point they see no relationship between these properties. Figures are recognized as having parts and are recognized by their parts. Usually, they know a list of properties, but they cannot decide which properties are necessary and which are sufficient to describe the object. Interrelationships between figures are still not seen, and definitions are not yet understood at this level.

Level 3: Abstraction

At level 3 students perceive relationships between properties and between figures, they are able to establish the interrelationships of properties both within figures (e.g., in a quadrilateral, opposite angels being equal necessitates opposite sides being equal) and among figures (a rectangle is a parallelogram because it has all the properties of a parallelogram). So, at this level, class inclusion is understood, and definitions are meaningful. They are also able to give informal arguments to justify their reasoning. However, a student at this level does not understand the role and significance of formal deduction.

Level 4: Deduction

The 4th level is the level of deduction: students can construct smaller proofs (not just memorize them), understand the role of axioms, theorems, postulates and definitions, and recognize the meaning of necessary and sufficient conditions. The possibility of developing a proof in more than one way is also seen and distinctions between a statement and its converse can be made at this level.

Level 5: Rigor

This level is the most abstract of all. A person at this stage can think and construct proofs in different kind of geometric axiomatic systems. So, students at this level can understand the use of indirect proof and proof by contra-positive and can understand non-Euclidean systems.

Note that level 5 is questioned to be a real level. Fortunately, the abstraction of level 5 does not occur in high school. Still, we added it to the list for completion. We do not discuss level 5 in *Table 1*.

Description of level	**NCC (2020)**	**FC**
Level 1 1. Use of imprecise properties (qualities) to compare drawings and to identify, characterize, and sort shapes. 2. References to visual prototypes to characterize shapes. 3. Inclusion of irrelevant attributes when identifying and describing shapes, such as orientation of the figure on the page. 4. Inability to conceive of an infinite variety of types of shapes.	1-4. classes „distinguishes and separates collected or created figures by freely chosen or geometric properties; observes the common properties of the figures, finds corresponding labels of given sets of figures; finds the figures without common properties" [2, p.334.] „recognizes the polygons from plain figures; name the triangles, quadrilaterals, circles" [2, p.334.]	1-2. classes 1. „Name the polygons by the numbers of the sides and the vertices." 2. „pick polyhedral and sphere form given figures by senses" 4. „Recognize, choose and name the triangles, quadrilaterals and circles." 5. „Search, notice and name the typical properties of solids: plain or curve face, with or without holes", Search, notice and name the typical properties of

5. Inconsistent sortings; that is, sortings by properties not shared by the sorted shapes. 6. Inability to use properties as necessary conditions to determine a shape; for example, guessing the shape in the mystery shape task after far too few clues, as if the clues triggered a visual image.		edges straight or curve borderlines, "holey" 6. „Free choice of given or constructed solids" [5, p.24.-25.]

Level 2 1. Comparing shapes explicitly by means of properties of their components. 2. Prohibiting class inclusions among general types of shapes, such as quadrilaterals. 3. Sorting by single attributes, such as properties of sides, while neglecting angles, symmetry etc. 4. Application of a litany of necessary properties instead of determining sufficient properties when identifying shapes, explaining identifications, and deciding on a mystery shape. 5. Descriptions of types of shapes by explicit use of their properties, rather than by type names, even if known. For example, instead of rectangle, the shape would be referred to as a four-sided figure with all right angles. 6. Explicit rejection of textbook definitions of shapes in favor of personal characterizations. 7. Treating geometry as	1-4. classes „name the shape of the faces of the cuboid, recognises the congruent faces of the cuboid; can distinguish the edges and the vertices of the cuboid; knows the number of faces andvertices of a rectangle, can show the equality of the angles of the rectangle with folding; points at the sides of a rectangle of equal length, decribes their mutual realtionship, shows and counts the diagonals and axis of symmetry of the rectangle; observes the properties of the cube as a special cuboid, and the properties of the square as a special rectangle; names the cuboid, the cube, the rectangle, and the square based on their observed properties" [2, p.334.]	3-4. classes 1. „selecting plain figures or solids by their joint properties" 2. „Naming common protperties os solids and plain figures by htir properties in common, observing, labelling" ** 3. „separating object into sets by properties in common" * 4. „Naming common protperties of solids and plain figures by their properties in common, observing," ** 5. „Describing polygons with properties in common, recognized by themselves" 6. „Desrcibing solids and plain figures by their geometric properties" 7. „Naming common protperties os solids and plain figures by htir properties in common, observing, labelling"** 8. „Choose the square from the rectangles by its sides and symmetries" [5, p.54.]

physics when testing the validity of a proposition; for example, relying on a variety of drawings and making observations about them. 8. Explicit lack of understanding of mathematical proof.		

Level 3 1. Formation of complete definitions of types of shapes. 2. Ability to modify definitions and immediately accept and use definitions of new concepts. 3. Explicit references to definitions. 4. Ability to accept equivalent forms of definitions. 5. Acceptance of logical partial ordering among types of shapes, including-class inclusions. 6. Ability to sort shapes according to a variety of mathematically precise attributes. 7. Explicit use of "if, then" statements. 8. Ability to form correct informal deductive arguments, implicitly using such logical forms as the chain rule (if p implies q and q implies r, then p implies r) and the law of detachment (modus ponens). 9. Confusion between the roles of axiom and theorem.	5-8. classes „knows the special quadrilaterals: trapezoids, parallelograms, rectangles, kites, rhombuses, inscribed trapeze, square; (…) recognizes the congruent figures in the plain (…) knows the properties of the triangles: the sum of the interior angles, the sum of the exterior angles, triangle inequality; knows Pythagoras' theorem and applies it in calculation tasks; knows the properties of quadrilaterals: the sum of the interior angles, the sum of the exterior angles, difference between convex and concave, the concept of diagonal" [2, p.337.]	5-8. classes 1. „recognize the special quadrilaterals (trapezoids, parallelograms, rectangles, kites, rhombuses, inscribed trapeze, square) and determine their major properties by an illustration; application; set diagram"*** 3. „Knows and applies the properties of quadrilaterals: the sum of the interior angles, the sum of the exterior angles, difference between convex and concave, the concept of diagonal" 5. „recognize the special quadrilaterals (trapezoids, parallelograms, rectangles, kites, rhombuses, inscribed trapeze, square) and determine their major properties by an illustration; application; set diagram" *** 6. „recognize the special quadrilaterals (trapezoids, parallelograms, rectangles, kites, rhombuses, inscribed trapeze, square) and determine their major properties by an illustration; application; set diagram" *** 7. „Knows and applies the Pythagoras' theorem." [5, p.28.]

Level 4 1. Clarification of ambiguous questions and rephrasing of problem tasks into precise language. 2. Frequent conjecturing and attempts to verify conjectures deductively. 3. Reliance on proof as the final authority in deciding the truth of a mathematical proposition. 4. Understanding of the roles of the components in a mathematical discourse, such as axioms, definitions, theorems, proof. 5. Implicit acceptance of the postulates of Euclidean geometry." (Burger, Shaughnessy, 1986)	9-12. classes „knows and applies notions and theorems concerning special points, circles and lines of triangles; knows and applies the Pythagoras' theorem and reversal of the theorem; (…) knows and applies Thales' theorem and reversal of the theorem (…) knows and applies the theorems of the proportion of the perimeter and area of similar plain figures; (…) knows and applies the theorems of the proportion of the surface area and volume of similar solids." [2, p.340-341.]	9-10. classes 1. „The area of the regular polygon by dissection" **** 2. „The area of the regular polygon by dissection" **** 3. „Prove the theorem concerning the intersection of the side bisectors and the angle bisectors" (…) Prove Pythagoras' theorem" „Knowing and applying theorems on convex polygons: number of diagonals, sum of interior and exterior angles." „Prove Thales' theorem" 4. „knows and applies notions and theorems concerning special points, circles and lines of triangles: side and angle bisectors, midlines, weight lines, altitudes, inscribed and outscribed circles." 5. „Execute basic Euclidean constructions with conventional or digital equipment: segment and angle bisector, orthogonal and parallel lines, copying angles" [5, p.13-16.]

Table 1.

It is not difficult to observe that both the NCC and the FC follow the Van Hiele levels. Indeed, with very few exceptions the correspondence is a word-by-word match.

Table 1. shows a matching between the Van Hiele levels and the topics of the NCC and the FC. Level 1 is reached at grade 2, level 2 at grade 4, level 3 at grade 6 and level 4 at grade 10. For comparison, we have quite a change to the levellings of 2014. In Muzsnay at al. (Muzsnay & al., 2020) the access to the corresponding levels is the following: Level 1 at grade 4, level 2 at grade 6, level 3 at grade 8, and level 4 at grade 10. We tried to

number the items of the framework by the numbers of the levels such that the same numbered element of the framework explains the corresponding item of the levels. We do not go through all items step by step. Quite a number of them is straightforward. We examine a few crucial items on every level. Keep in mind that the description of the levels is usually general about understanding, while the NCC and FC is specific. This is so, because the curriculum is supposed to be concrete. Consider, for example, the use and understanding a theorem. The use of Pythagoras' theorem is required in grade 5-8 and complete understanding of it comes in grade 9-12. The distinction is made by Level 3 item 7. Explicit use of "if, then" statements, and Level 4 item 3. "Reliance on proof as the final authority in deciding the truth of a mathematical proposition". Also, the theorems listed in grade 9-10 assume level 3 item 4. Understanding of the roles of the components in a mathematical discourse, such as axioms, definitions, theorems, proof. This rejects Level 2 item 9. Confusion between the roles of axiom and theorem. Hence a grade 9-10 pupil is not supposed to be on level 3 about handling a statement, and has to reach level 4.

Now we discuss the items of the colomns to argue that our table shows a parallelism. The borderlines between levels and items are not sharp, as it is not supposed to be. On Level 2 item 5 corresponds straightforwardly to items 5 and 6 of the framework, but item 6 also describes item 6 of level 1, as well. Item 7 of the framework relates to both item 4 and 7 of the level descriptions. To see that we do not expect more can be read from item 6 of level 2.

There is an almost perfect match between description of level 3 and the framework. The only non-explained item is item 9. Indeed, a framework does not tell what not to know. The lack of distinction between the parts of foundation of geometry can be concluded from the description of level 4. For example, in the proofs of Thales' theorem and the theorems on angles involves implicit use of Euclid's axioms, and no knowledge in grade 5-8 refers to them.

There are some cases where the correspondence is not word by word, but still obvious. Some explanation should be given to level 4. Level for asks for general competencies, while FC gives a list of theorems and their applications. The list of theorems in FC is so long that we can assume that items 2 and 3 are present. Also, not involving only geometry the following general requriments are in NCC, 2020 for all topics. "… in grades 9-12 the deductive side of mathematics is accentuated." Or: „Forming new notions and purposefully finding and discovering new problem solving techniques developes the competencies of mobilizing knowledge and using adequate problem solving strategies.“ or „forms conjectures and verifies them using steps of formal logic“ or „can argue for simple statements and can prove theorems.“ finally: „can construct plan of solution or an algorithm to solve a given problem.“

These elements of NCC with the items in Table 1. fully cover Level 4 of the Van Hiele scale.

Final exam

Following the lead of (Muzsnay & al., 2020) we now compare the level needed for solving the final exam of geometry problems to level 4 to be achieved by a grade 12 student. In (Muzsnay & al., 2020) they showed as examples the following problem. This problem was told to be a more difficult geometry problem form an advanced level final exam.

A motion sensor is on the top of a 4 m high vertical pole. The lamp connected to the sensor illuminates vertically downwards at a rotational cone of 140°.

a) Make a sketch with the details.
b) How far is the farthest illuminated point from the lamp?
c) Does the sensor lamp illuminate an object on the ground 15 m from the bottom of the pole?
d) There are hooks on the pole, one per meter, in order to hang the motion detector lamp. Which hook should we use in order that the lamp illuminates at most $100\ \mathrm{m}^2$ on the horizontal ground? (Numbering of the hooks starts from the bottom of the pole.)

(2006, basic level, Task 18, k_mat_06maj_fl.pdf)

The structure of the final exam is different. Here we present a geometry problem of the final exams in 2018 and 2019 (Emelt szintű feladatsor 2018, 2019).

1. The length of the sides of a triangle are 7 cm, 9 cm and 11 cm.

a) Prove that it is an acute-angled triangle.

The length in centimetres of the sides of a right triangle are three consecutive members of an arithmetic progression.

b) Prove that the ratio of the sides of the triangle is 3:4:5.

c) The area of this triangle is $121{,}5\ \mathrm{cm}^2$. Calculate the length of the sides of this triangle.

(2018. advanced level, Task1, e_mat_18maj_fl.pdf)

By the official solution one has to calculate either the cosine of the largest angle or the cosine of all angles. So, to solve 1 a) a student needs to substitute the data into a known formula, the law of cosines. If this formula hits her/his mind, a rather simple calculation is enough to find the answer. This is a maximum 3 of Van Hiele levels, no proof, no sequence of ideas, no complex application of theorems, definitions and axioms are needed.

Solution 1. b)		
Let denote by $a-d, a, a+d$ the sides of the triangle ($0<d<a$)	1 point	$b, b+d, b+2d$, $(b,d>0)$
By Pythagoras' theorem $(a-d)^2+a^2=(a+d)^2$.	1 point	$b^2+(b+d)^2=(b+2d)^2$
After squaring and rearrangement: $a^2=4ad$	1 point	$b^2-2db-3d^2=0$
(Dividing by $a\neq 0$) $a=4d$	1 point	Solving for b with the quadratic formula we obtain $b=3d$ ($b=-d$ is not a solution)
Hence the sides of the triangle are $3d, 4d$ és $5d$, and their proportion is 3:4:5 as we wanted.	1 point	
Total	5 points	

Table 2

For 1 b) the knowledge of Pythagoras' theorem suffices. No more geometry is involved. This is just a simple application of a formula again, and we would say that this is not even an application of an "if, then" statement. Formally, it is because Pythagoras' theorem is applied. But Pythagoras' theorem is so common, so wellknown that one needs not to think too much to apply.

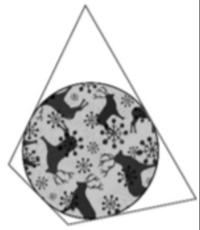
We do not claim that the problem is not an appropriate problem and we are far from claiming that we should not ask Pythagoras' theorem. The only thing we argue for is that the solution of this problem does not require Level 4.

Solution 1. c)		
The area of the triangle is $\frac{3d\cdot 4d}{2}=121{,}5$.	1 point	
Then $12d^2=243$, and (because of $d>0$) $d=4{,}5$.	1 point	
Thus, the lengths of the sides are 13,5 cm, 18 cm and 22,5 cm.	1 point	
Total	3 points	

Table 3.

This solution is based on the formula of the area of a right triangle. No real Van Hiele level is needed to solve. The area of a right triangle is learnt in grade 6.

7. In preparation for the school Christmas market, the kids made different types of decorations so that they cut around circular pictures printed on coloured cardboard with four straight cuts each. The length of the sides of one of these tangential quadrilateral decorations are four consecutive members of an arithmetic progression (in some order). The length of a side of this quadrilateral is 23 cm, and the perimeter of this quadrilateral is 80 cm.



c) What can be the length of the last three sides of that quadrilateral?

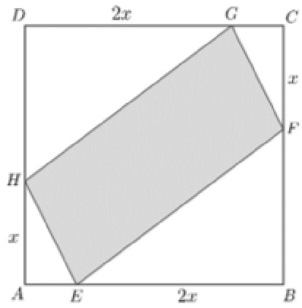
Remark: The questions a) and b) of problem 7 are not geometry related. (2018. advanced level, Task 7, e_mat_18maj_fl.pdf)

Second solution 7. c)		
WLOG we may assume that the arithmetic sequence is increasing. In this case 23 is the 3rd or 4th element of the sequence, as it has two smaller and two greater elements then 20.	1 point	
If 23 is the 3rd element, then $(23-2d)+(23-d)+23+(23+d)=80$. The differencs of the sequence is denoted by d.	1 point	
Hence $92-2d=80$ and $d=6$.	1 point	
If 23 is the fourth element of the sequence, then $(23-3d)+(23-2d)+(23-d)+23=80$.	1 point	
Hence $92-6d=80$ and $d=2$.	1 point	
Thus, the other three sides are either 11, 17 and 29 or 17, 19 and 21 (cm). (In both cases there exists such a convex quadrilateral).	1 point	
As $11+29=17+23$ and $17+23=19+21$, both are tangential quadrilaterals.	1 point	
Total	7 points	

Table 4.

The solution barely uses any geometry knowledge. Geometry occurs in the last step, when the solver has to check that the sums of the opposite sides are equal. Note that for any arithmetic progression the sum of two consecutive elements is equal to the sum of the previous and next elements. Also, the solution does not require to state the theorem that a quadrilateral is tangential if and only if the sum of the lengths of the opposite sides are equal. Hence, we can say that this solution is not asking for Level 4. Out of the maximal 7 points 1 point is for geometry.

1. The length of the sides of the $ABCD$ square is 4 m. We inscribe the $EFGH$ parallelogram into the square as shown on the illustration. The length of the AH segment is x metre, and also, the length of the CF segment is x metre. The length of the BE segment is $2x$ metre, and also the length of the DB segment is $2x$ metre $(0 < x < 2)$



a) Prove that the area of the inscribed parallelogram (in m^2):

$$T(x) = 4x^2 - 12x + 16.$$

b) Calculate the value of x so that the area of the inscribed parallelogram is minimal.

c) Calculate the angles of the inscribed parallelogram, if $x = 1{,}25$.

(2019. advanced level, Task1, e_mat_18maj_fl.pdf)

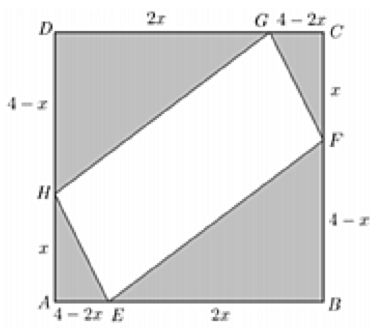
<table>
<tr><td colspan="3">Solution 1. a)</td></tr>
<tr><td rowspan="3">
</td><td>(The area of the parallelogram can be obtained by subtracting the sum of the areas of the four triangles.)
$BF = DH = 4 - x$
and
$AE = CG = 4 - 2x.$</td><td>1 point</td></tr>
<tr><td>$T(x) = 16 - 2 \cdot \frac{x(4-2x)}{2} - 2 \cdot \frac{2x(4-x)}{2}$</td><td>1 point</td></tr>
<tr><td>$T(x) = 16 - 4x + 2x^2 - 8x + 2x^2$</td><td>1 point</td></tr>
<tr><td colspan="2">Rearranging we obtain $T(x) = 4x^2 - 12x + 16.$</td><td>1 point</td></tr>
<tr><td colspan="2">Total</td><td>4 points</td></tr>
</table>

Table 5.

For 1 a) the student has to recognize the triangles and the parallelogram on the picture. This is Level 3 by definition. Then the formulas for area are applied, which requires no more geometrical argumemts. Part b) belongs to algebra or analysis, it has two sample solutions, one of them with derivatives. Part c) calculates the angles finding first the tangents of the corresponding angles, use of Level 3 is required by definition.

4. A conjurer uses two uniform “four-sided dice” in his show. The shape of the four-sided dice is a three-sided pyramid, which has 6 cm long base edges, and whose side edges and base sheet make an angle of 30°. a) Calculate the volume of the pyramid! (2019. advanced level, Task 4, e_mat_18maj_fl.pdf)

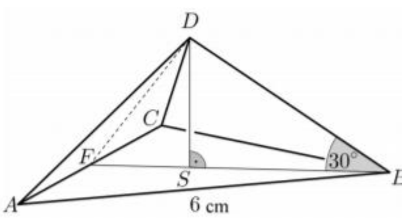
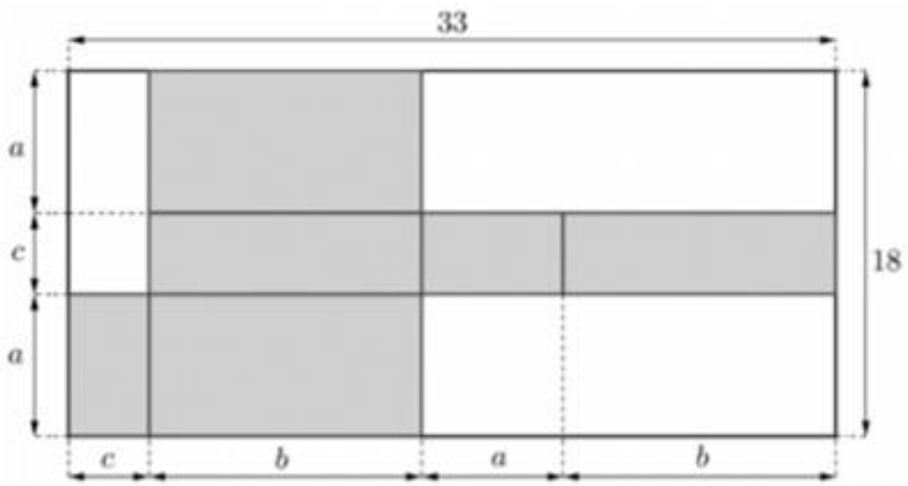
Solution 4. a)		
	Use the notations of the figure. The centre of mass of the ABC face of the pyramid is S. DS is perpendicular to the base and by the conditions $SBD\angle = 30°$.	1 point
	BS is the two third of the height of the triangle ABC. $BS = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \cdot 6 = 2\sqrt{3} (\approx 3{,}46)(\text{cm})$	2 points
The height of the pyramid is $DS = BS \cdot \text{tg}30° = 2$ (cm).		1 point
The area of ABC is $T = \frac{AB^2 \cdot \sqrt{3}}{4} = \frac{6^2 \cdot \sqrt{3}}{4} = 9\sqrt{3} (\approx 15{,}59)(\text{cm}^2)$		1 point
The volume of the pyramid is $V = \frac{T \cdot DS}{3} = 6\sqrt{3} (\approx 10{,}4)\text{cm}^3$.		1 point
	Total	6 points

Table 6.

The first part of the problem is to find a planar section of the pyramid such that we can calculate its height. For this one should recognize the section determined by an edge and the altitude. Recognizing parts of these objects is Level 3. The section obtained results the half of a regular triangle such that its altitude is known. So, we need the formula for the length of the altitude in a regular triangle. This altitude is calculated from another altitude of an other regular triangle. As analyzing the parts of the pyramid one has to observe that the altitude of the sectional triangle is two third of the altitude of the base. This requires to recognize that in the regular triangle the center divides the altitude in 1:2. Then we use an “if, then” statement, to find the appropriate length. The description of these steps looks more complicated than the solution itself, but is necessary to understand that no step belongs to Level 4. The only chance for using

Level 4 is finding the relationships between the lengths, but the regular triangle is so special, and its properties differ so much from a triangle in general that we claim that no Level 4 is necessary to solve the problem.

5. Someone makes a cuboid-shaped box from a cardboard sheet (with cut-out and folding). The area of the cardboard sheet is 33 x 18 cm. They choose the mesh and the sizes of the box as you can see it on the illustration (the dark-coloured part).



Determine the volume of the box, if $a = 7$ cm.

How to choose the length of the sides a, b, c so that the volume of the box be as big as possible?

Any three vertices of cuboid determines a triangle. How many triagles are there such that its vertices are vertices of the cuboid and it does not lie not on the plane of any of the faces?

(2019. advanced level, Task 5, e_mat_18maj_fl.pdf)

Solution 5. a)		
(Counting in centimetres) using $2a + c = 18$ we get $c = 18 - 2 \cdot 7 = 4$.	1 point	
By $a + 2b + c = 33$ we get $b = \frac{33-7-4}{2} = 11$.	1 point	
The volume of the cuboid is $abc = 7 \cdot 11 \cdot 4 = 308$ cm^3.	1 point	
Total	3 points	

Table 7.

In part a) the elements of the plain figure have to be recognized then after a small calculation the formula for the volume gives the answer. This is probably lower Level 3.

Part b) is an application of the formula for the volume of the cuboid. No extra geometry knowledge is applied in its solution.

Solution 5. c)		
The 8 vertices determine $\binom{8}{3} = 56$ triangles.	1 point	
We have to subtract the number of those for which their plane coincides with one of the faces of the cuboid. There are four such triangles on each face.	2 points	
The number of triangles is $(56 - 24 =)$.	1 point	
Total	4 points	

Table 8.

In part c), out of the recognition of a counting argument the analysation of the parts of the figure is needed. Indeed, the leading geometry knowledge is that any three non-linear points determines a triangle. Then, we need to subtract the number of those on a face. Level 4 is not involved in these arguments.

Solution 6. a)		
The perimeter of the triangle is 30 units. Late the sides be denoted by x, x and $30 - 2x$		1 point
By the deviation $$\sqrt{\frac{(10-x)^2 + (10-x)^2 + (2x-20)^2}{3}} = 3\sqrt{2}.$$		1 point
$200 - 40x + 2x^2 = 18$ $2x^2 - 40x + 182 = 0$ $x^2 - 20x + 91 = 0$	$\sqrt{2(10-x)^2} = 3\sqrt{2}$ $\sqrt{(10-x)^2} = 3$ $\lvert 10 - x \rvert = 3$	1 point
$x = 7$ or $x = 13$		1 point
The sides of the triangle are $7, 7, 16$ in the first case and $13, 13, 4$ in the second case.		1 point
Discussion: the first case cannot occur, because the triangle inequality does not hold. The second case is possible, the triangle inequality holds. (The deviation is $\sqrt{\frac{3^2+3^2+6^2}{3}} = \sqrt{18} = 3\sqrt{2}$.)		1 point
Total		6 points

Table 9.

Going through the calculations two informations from geometry is required: an isosceles triangle has two equal sides and the triangle inequality. These belong to levels 2 and maybe 3, when we apply the inequality.

Van Hiele levels of presevice math teacher students:

In (Bereczky-Zámbó & al., 2018) the Van Hiele level of the first year presevice math teacher students were tested. The students filled in the test in February 2015. This was the last year before introducing the new final exam. They filled it in right before their first geometry course started in their 2nd semester. The results can be seen in Table 10. The table contains the strong Van Hiele levels. It can be seen that 22 students out of 46 were on level 5 and 24 on level 3. This means that 22 students were on the level of grade 8 determined by the NCC and 24 of them on the expected level 5.

We have run the same test the same part of the year for the first year peservice math teachers. The tests were evaluated on the strong way, as before, the results can be seen in Table 10. In 2021 out of 40 in 40 students 10 reached level 5, which is 25% and 13 reached at least level 4 which is 33%. Both percentages can be compared to the 48% in 2015. There are students below level 3, 18 % of them, that did not happen before. This is a fact that we have to think about.

	Van Hiele level				
	5	4	3	2	under 2
2015	22	0	24	0	0
2020	10	3	20	2	5

Table 10.

Conclusion

In this paper the level of geometry education in mathematics education in Hungary is considered. We investigated the relationship between the National Core Curriculum, the Framework Curiculum and the final exam from the geometry point of view. We used the van Hiele levels as a tool for comparison. The theory of the Van Hieles is a widely accepted measure for understanding geometry worldwide. We found that the the NCC and the FC step by step follow the Van Hiele levels in Hungary. At grade 10 pupils are to achieve level 4. Then we took the final exams of years 2018. and 2019, and thoroughly went through all geometry problems. We found by the official sample solutions that none of the problems requires knowledge, skills and competencies above level 3. This means that the geometry knowledge required in the final exam is lower than the one prescribed by the NCC or FC. In (Kovács, 2017) the authors argued that the by its nature the final exam is predictable and has a high impact on the actual curriculum and teacher activities in class. Indeed, this exam also serves as an entrance exam for the tertiary education, as well. The scores on the final exam have a crucial role only in the students' lives. Teachers at school are evaluated by the success of their students on the final exam. Hence it is a common interest of all parties to prepare for the final exam. This way there is no straightforward motivation to teachers to teach students to Level 4. They rather concentrate on parts of mathematics needed for the final exam.

According to (Muzsnay & al., 2020) in 2015 the gap between the final exam's requirement and the NCC's requirement indicates further problems for higher education. They claim that there is a big difference between the geometry knowledge of students entering the university and the knowledge required by the universities. By our survey this gap has just widened since then. The Van Hiele level of preservice math teacher students has declined. The expected geometry knowledge of students finishing hihgschool is level 4. Earlier, in 2015, 48% of preservice math teacher students reached at least level 4, now this number is 33%. The difference is significant. We do not claim that it is the fault of the final exam, and we do not claim that this is a fault of the highschool math education. Still, it is a possibility that we have to consider. Aiming for a good final exam might distort the aim for a higher knowledge, even if it is required by the NCC.

When Ptolemy 1st Soter, king of Egypt, found Euclid's seminal work, the Elements, too difficult to study, he asked Euclid to show him an easier way to master it. Euclid's answer to the king is still valid:

„There is no royal road (short cut) to geometry."

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